ADDITIONS AND CORRECTIONS TO "ON A CONVEXITY CONDITION IN NORMED LINEAR SPACES"

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Except as noted, all references below are to the above named paper in the Transactions of the American Mathematical Society, Vol. 125, pp. 114–146.

1. **Errata.**

p. 115 l. 7b. For the second "on" read "of".

p. 120 l. 7b and 5b. For "$j=1$" read "$j=0$".

p. 120 l. 3b and 2b. Replace the lines by the following:

$$\text{"} \leqq \frac{k(k-1)}{k(1-\varepsilon)-1}(1-\varepsilon)^{m+1} \quad \text{since } k(1-\varepsilon) > 1$$

$$\leqq K(1-\varepsilon)^{\log_k n} \quad \text{since } \log_k n \leqq m+1\text{".}$$

I thank Mr. Peter Warren for calling to my attention the error corrected by these last two lines.

2. **Additions to** IV.2. We show below that if $\mathfrak{X}$ and $\mathfrak{Y}$ are infinite-dimensional NLS's and if $\mathfrak{X}$ is a conjugate space (in particular, if $\mathfrak{X}$ is reflexive), then $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ is not B-convex. This implies that if the conjecture that every B-convex space is reflexive is true, then also the conjecture that every $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ is not B-convex for infinite-dimensional $\mathfrak{X}$ and $\mathfrak{Y}$ is true (since, by IV.2, if $\mathfrak{X}$ is not B-convex, then $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ is not B-convex).

The principal tool here is the theorem of Aryeh Dvoretzky (see, e.g., "Some results on convex bodies and Banach spaces," Proceedings of the International Symposium on Linear Spaces, pp. 123–160, Pergamon Press, New York, 1961) which states that every infinite-dimensional Banach space contains arbitrarily good approximations of finite-dimensional Hilbert space of every finite dimension.

Let $a_n(\cdot)$ be the sequence of period 2^n which starts with 2^{n-1} $(+1)$'s and then 2^{n-1} (-1)'s. Let k be fixed $\geqq 2$ and let $m=2^k$. Denote the usual basis of l_2 by $\{\delta_n\}$. Fix $\varepsilon>0$ and use Dvoretzky's theorem to find $x_1, \dots, x_m$ of unit norm in $\mathfrak{X}$ such that

$$(1-\varepsilon)\left\|\sum_{i=1}^{m} \alpha_i x_i\right\| \leqq \left(\sum_{i=1}^{m} \alpha_i^2\right)^{1/2}$$

for all scalars α_i. For $j=1, \dots, k$ define $T_j\colon l_2 \to \mathfrak{X}$ by $T_j(\delta_i)=(1-\varepsilon)a_j(i)x_i$ if $1 \leqq i \leqq m$ and $T_j(\delta_i)=0$ otherwise, and extend by linearity and continuity. Then

$T_j \in \mathfrak{B}(l_2, \mathfrak{X})$ and $\|T_j\| \leqq 1$. By the usual trick (see I.3(iv), the proof of Theorem II.3, or IV.2)

$$\| \pm T_1 \pm T_2 \pm \cdots \pm T_k\| \geqq k(1-\varepsilon)$$

for all choices of the $+$ and $-$ signs. So $\mathfrak{B}(l_2, \mathfrak{X})$ is k, ε-convex for no $k \geqq 2$ and $\varepsilon > 0$, hence is not B-convex.

Since the adjoint mapping of $\mathfrak{B}(l_2, \mathfrak{X})$ into $\mathfrak{B}(\mathfrak{X}^*, l_2)$ is an isometry, it follows that the latter space is not B-convex.

Finally, for $k \geqq 2$ and $\varepsilon > 0$, pick $T_1, \ldots, T_k$ in $\mathfrak{B}(\mathfrak{X}^*, l_2)$ of unit norms so that

$$\| \pm T_1 \pm T_2 \pm \cdots \pm T_k\| \geqq k(1-\varepsilon)$$

for all choices of the $+$ and $-$ signs. By considering the image of points where these 2^k linear combinations of T's nearly achieve their norms we find a projection P of l_2 onto a finite-dimensional subspace such that

$$\| \pm PT_1 \pm PT_2 \pm \cdots \pm PT_k\| \geqq k(1-2\varepsilon)$$

for all choices of the $+$ and $-$ signs. Again using Dvoretzky's theorem, we find a linear map $S\colon P(l_2) \to \mathfrak{Y}$ which has norm 1 and is so nearly an isometry that

$$\| \pm SPT_1 \pm SPT_2 \pm \cdots \pm SPT_k\| \geqq k(1-3\varepsilon)$$

for all choices of the $+$ and $-$ signs. Since each SPT_j is an element of $\mathfrak{B}(\mathfrak{X}^*, \mathfrak{Y})$ of norm at most 1, we see that $\mathfrak{B}(\mathfrak{X}^*, \mathfrak{Y})$ is not k, 3ε-convex, and since k and ε were arbitrary, it is not B-convex.

Other conditions can be given to assure non-B-convexity of $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$. For example, if $\mathfrak{X}$ has a family $\{\mathfrak{X}_n\}$ of finite dimensional subspaces including spaces of arbitrarily large dimension such that each $\mathfrak{X}_n$ is symmetric (has a basis such that the reflections in the coordinate hyperplanes are isometries), if there is a uniformly bounded family $\{P_n\}$ of projections on $\mathfrak{X}$ such that the range of P_n is $\mathfrak{X}_n$, and if for each n there is an isomorphism $T_n\colon \mathfrak{X}_n \to \mathfrak{Y}$ such that the families $\{T_n\}$ and $\{T_n^{-1}\}$ are uniformly bounded, then a construction like the preceding together with Lemma I.4 shows that $\mathfrak{B}(\mathfrak{X}, \mathfrak{Y})$ is not B-convex. Similarly, if $\mathfrak{X}$ has an infinite-dimensional direct summand which is a conjugate space, then $\mathfrak{B}(\mathfrak{X},\mathfrak{Y})$ is not B-convex.

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